ON PLASTICITY LAWS FOR WORK-HARDENING MATERIALS*

(O ZAKONAKH PLASTICHNOSTI DLIA MATERIALA S UPROCHNENIEM)

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This study is an attempt to trace the development of methods used in solving the problem of stress-strain relations in work-hardening plastic materials.

This review does not deal with problems of large deformations, thermodynamics of deformations, or questions related to time-dependent properties of materials. Neither does it discuss methods of solving boundary-value problems on the basis of this or that plasticity law. Thus, while remaining within a fairly narrow scope of problems, we were as much as possible attempting to describe that logical path of the evolution of plasticity theory, which in a reasonably short period of time led researchers from the simple Hencky-Nadai theory to the contemporary, rather broad concepts. For the sake of a continuous description, we have sometimes found it necessary to sacrifice the mere history of the problem.

In order not to obstruct this presentation with unnecessary details, we will only describe contributions which we consider to be the more important ones. Moreover, it should be noted that in reference to these studies we found it sometimes convenient to change somewhat the form of presentation of the particular source, without, of course, changing the essence of the material.Specifically, in all instances where the original source describes tensorial relations, we are considering the relationship between the corresponding deviators. This interchange cannot cause any misunderstandings, since an independence of small plastic deformations from the median pressure is usually assumed, and at the same time a consideration of the immediate relation between the deviators is more convenient, especially when a vectoral presentation of tensors is used.

The origin of the plasticity theory of an initially isotopic material with strain hardening is contained in the most simple mathematical theory of the 1920's. At that time, after the appearance of a number of papers,

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among which the leading place was occupied by the works of Hencky [1] and Nadai [2], the following hypothetic law of relation between stress and strain for elastic-plastic material was formulated:

$$\begin{aligned} S_{ij} &= 2G_s Э_{ij}, \quad dI_2 > 0, \\ dS_{ij} &= 2G dЭ_{ij}, \quad dI_2 \leqslant 0, \end{aligned} \qquad I_2 = \frac{1}{2}\sum S_{ij}^2 = \frac{1}{2} S_{ij} S_{ij} \tag{1}$$

where S_{ij} is the stress deviator, $Э_{ij}$, the strain deviator, G_s the secant modulus of the curve illustrating the dependence of the shear stress from shear strain obtained in a pure shear experiment, G is the elástic shear modulus. Further, it was determined by various experiments, among which those performed by Bridgman [3] were the most fundamental ones, that in small deformations

$$\sigma = Ke \tag{2}$$

with a high degree of accuracy, where σ is the mean or hydrostatic pressure, K is the elastic proportionality coefficient, e is the dilatation. Precisely the circumstance that the volume change is elastic, permitted to reduce the task of determining the relation between stress and strain to the task of establishing relations between the deviators of stress and strain, which is what was done in the law (1).

Lacking experimental data on work-hardening materials under combined stress, the only guidance available was intuition and, where possible, drawing of inferences from such well-known properties as described by the generalized Hooke's law and the curve of simple tension-compression or torsion of plastic work-hardening materials. Hence, it is understand able that the above law (1) differs from the elastic law only insofar that instead of the constant modulus A a variable coefficient G_s is introduced. More cannot be deduced from generalized Hooke's law, and the remainder of the formula (1) is essentially a generalization of the properties of such a curve. Properties of such a curve with respect to stress-strain relations can be expressed by

$$\sigma_x = E_s \varepsilon_x$$

during a continuous increase of stresses, an by

$$d\sigma_x = E d\varepsilon_x$$

during a decrease of stresses, following their continuous increase.

Thus, the increase or decrease of stress σ_x can serve as criterion for the selection of one or another relation. It could be stipulated that when $d\sigma_x > 0$, the first relation is to be used, and when $d\sigma_x \leqslant 0$, the second. However, a consideration of the negative values of σ_x introduces a correction into this criterion, as it is easily understood that for simple tension-compression the sign of the expression $\sigma_x d\sigma_x = \frac{1}{2} d(\sigma_x^2)$ will serve as a general criterion. Hence, the sign of the quantity $\sigma_x d\sigma_x$ in simple tension-compression serves as criterion of whether the material is governed by the law of the curve or by the elasticity law, i.e. serves as a loading criterion.

Generalizing the expression $\sigma_x\, d\sigma_x$ for complex loading, it would seem natural to replace it by $\sigma_{ij} d\sigma_{ij}$ where σ_{ij} is the stress tensor. But this would not be correct, since $\sigma_{ij} d\sigma_{ij} = S_{ij} dS_{ij} + 3\sigma\, d\sigma$ and already then [3] it was known, that the mean pressure σ does not influence plasticity in small deformations. It follows that without contradicting the experiment in uniaxial loading, the value (or to be more exact, the sign of the value) $S_{ij} d\, S_{ij} = dI_2$, can be, for all general purposes, taken as the loading criterion, which is precisely what we see in the Hencky-Nadai law.

Simplicity of the Hencky-Nadai law attracted numerous researchers, and based on that law it was possible to develop a fairly simple process of successive approximations (Il'iushin's method [4]) for boundary value problems. At the same time it became necessary to verify the Hencky-Nadai law experimentally. A number of articles was devoted to that subject (see series of articles in collection [5]). It was experimentally determined, that when a proportional increase of stress takes place at every point of the body, then the Hencky-Nadai law, as a whole, satisfactorily corresponds to the experimental data. At the same time, however, small systematic deviations were observed, which (Lode's diagram) indicated some inadequacy of the law.

The essence of the Hencky-Nadai law is the condition of proportionality of stress and strain deviators. This postulate can be substituted by a more general one, namely that the strain deviator is a function of the stress deviator

$$\vartheta_{ij} = F_{ij}(S_{11}, S_{12}, \ldots, S_{21})$$

Based upon considerations of tensor dimensions, it can be shown [6] that this corresponds to the assumption

$$\vartheta_{ij} = F(I_2, I_3^2)\,[P(I_2, I_3^2)\, S_{ij} + Q(I_2, I_3^2)\, I_3 t_{ij}] \tag{3}$$

where

$$I_2 = \frac{1}{2} S_{ij} S_{ij}, \qquad I_3 = \frac{1}{3} S_{ij} S_{jk} S_{ke}, \qquad t_{ij} = S_{ik} S_{kj} - \frac{2}{3} I_2 \delta_{ij} \tag{4}$$

This generalization of the Hencky-Nadai law was made by Prager [7], who showed that an acceptance of this generalization eliminates the previously discussed inadequacy of the Hencky-Nadai law.

Both the Hencky-Nadai law, which is frequently called the law of small elastic-plastic deformations, as well as the Prager law are characterized by finite relations between stresses and strains. Later on such plasticity laws were called laws of deformation.

Laws of deformation are the simplest variety of plasticity laws. In a general case stresses and strains may be related by some integrodifferential operations. The simplest class of such general relations are the linear tensor relations, i.e. relations in which tensor matrices are related by some linear operators, Prager [8] noted that if we stay within the framework of linear tensor relations, every plasticity law can be represented in the general form:

$$L\,[(S_{ij})] = L'\,[(\vartheta_{ij})] \tag{5}$$

where $(S_{ij}), (\vartheta_{ij})$ are the deviator matrices and L and L' are linear, scalar operators.

While the Hencky-Nadai law is a special case of the relation (5), the law (3) cannot be represented in this form (5) and is thus an example of the simplest non-linear tensor relationship.

The preference which is given to the Hencky-Nadai law, compared to other plasticity laws which can be deduced on the basis of (5), is due to its simplicity and relatively good correspondence to experimental data. Moreover, in 1947 Il'iushin [9] reported that if a proportional increase of deviators of stress or strain takes place at every point of the body, then all plasticity laws that are deducible from (5) will coincide with the Hencky-Nadai law, and the latter, in the case of such loading, can be considered a general law in the class of linear tensor plasticity laws. This type of loading at a point we will call in the future simple loading.

To justify subsequent calculations on the basis of the Hencky-Nadai formula, conditions still had to be found that would produce a simple loading at every point of the body. This problem was partially solved by Il'iushin [10], assuming the incompressibility of the material both in the elastic and in the plastic ranges, as well as an exponential law of the uniaxial loading curve. It was proved that with the given limitations, the loading will be simple at every point of the body, provided the external forces increase in proportion to one parameter (simple loading theorem).

An important place among plasticity problems is occupied by such problems, for which the loading differs considerably from the previously mentioned simple loading process (stability problems, for example), so that the application of the Hencky-Nadai or of the Prager laws becomes unjustified. Even though combined loading tests conducted in those times offered, for all technical purposes, a sufficient basis of adequacy of the deformation theory, insofar as it did correspond with the test data, the applicability of the deformation theory to loading, which differed considerably from simple loading, was questionable in view of the fact that according to the deformation theory, a rapid change of stress results in an equally rapid change of deformations. This condition, while true for an elastic body, results in a contradiction of the basic principles of the mechanics of solids when applied to some types of loading in a plastic range. This was for the first time demonstrated by Handelman, Lin and Prager [11].

All explorations of the mechanical properties of solids, and particularly of plasticity properties, are always based on the supposition that small changes of the external influences will bring about small effects. Thus, if two identical bodies, with identical properties, are loaded by two methods which differ little from each other, then the deformations of these bodies at each given moment must, too, differ very little. Let us assume that at some point of the body a state of stress is reached, as a result of continuous loading, and further, that I_2 after subsequent loading at this point remains constant, i.e. $dI_2 = 0$. According to Hencky-Nadai, this is a limiting case differentiating between loading and unloading, and hence such loading may be called neutral. Generally speaking, no matter what the criterion of loading, a neutral loading is a loading where the quantity which expresses the loading criterion remains constant. It would be natural to expect from real materials in case of such loading that the laws of plasticity and elasticity would coincide. However, the truth of the matter is that not a single law of deformation theory satisfies this condition, which was termed the continuity condition. For simplicity purposes, we will, following Prager [12], consider here only the Hencky-Nadai law.

Assuming that the loading is neutral, we will rewrite the law (1) in a differential form:

$$dS_{ij} = 2G_s d\vartheta_{ij} + 2\frac{dG_s}{dI_2} dI_2 \vartheta_{ij}$$

Since $dI_2 = 0$, it follows that (1) at neutral loading will be

$$dS_{ij} = 2G_s d\vartheta_{ij}$$

On the other hand, the elasticity law stipulates that

$$dS_{ij} = 2G d\vartheta_{ij} \tag{7}$$

and since $G \neq G_s$, (6) and (7) do not coincide.

This situation could have been possibly avoided by selection of a different loading criterion. However, it turns out that the loading criterion is not arbitrary, but is determined by the very law of relations between stresses and strains. Obviously the work of stress in the plastic components of strain (total work minus work in elastic strain components) is irreversible, which is also true for the plastic deformation itself, which cannot be asserted for the total work. Therefore, in the plastic range the condition

$$dW^p \geqslant 0 \tag{8}$$

has to be satisfied, in which the equality sign is only possible in the trivial case when the increments of plastic deformations themselves equal zero (we are not considering perfectly plastic materials). Condition (8) is usually known as the irreversibility condition. Applying this condition to law (1), we will necessarily arrive at $dI_2 > 0$ as our loading criterion. Indeed, from (1) we find

$$d\partial_{ij}^{p} = d\partial_{ij} - d\partial_{ij}^{y} = \frac{1}{2}\left(\frac{1}{G_s} - \frac{1}{G}\right) dS_{ij} - S_{ij}\frac{dG_s}{G_s^2}$$

$$dW^p = \sum S_{ij} d\partial_{ij}^{p} = \frac{1}{2}\left[\left(\frac{1}{G_s} - \frac{1}{G}\right)\sum S_{ij} dS_{ij} - \sum S_{ij}^2 \frac{dG_s}{G_s^2}\right] =$$

$$= \frac{1}{2G}\left[\frac{G}{G_s} - 1 - \frac{2GI_2}{G_s^2}\frac{dG_s}{dI_2}\right] dI_2$$

and condition (8) requires that

$$\left[\frac{G}{G_s} - 1 - \frac{2GI_2}{G_s^2}\frac{dG_s}{dI_2}\right] dI_2 \geqslant 0 \tag{9}$$

in the plastic range. If, which is true for most materials, we assume that during loading

$$\frac{dG_s}{dI_2} > 0 \tag{10}$$

then the square bracket in (9) is positive and the irreversibility condition results in $dI_2 > 0$ during loading.

Thus, no other criterion (at least for materials that satisfy condition (10)) can be proposed for the Hencky-Nadai law and the last possibility to satisfy the condition of continuity is eliminated.

The same can be proved for a general case of the deformation theory as well. The continuity condition in any such law is not satisfied, and it was the search for a plasticity law which would satisfy the continuity condition that greatly influenced the development of the so-called flow theory or the incremental theory of deformation.

The above-mentioned work of Handelman, Lin and Prager should be considered as the initial step for the formulation of such a theory. It should be mentioned that it was discovered only later, that Laning had, as early as 1942, proposed in an unpublished paper the flow theory in its simplest form. While Laning's theory was a generalization of the Reuss [13] theory and hence based ultimately on hydrodynamic analogies, the theory that was proposed by Handelman, Lin and Prager was called flow theory only by inertia and was derived from other considerations. It should have been more correctly called a theory of increments of deformations or the differential theory of plasticity.

The derivation of stress-strain relations incorporated in the last paper can be interpreted as follows. It is assumed that: (1) the increment of the strain deviator is fully determined by the stress deviator and its increment; (2) this relation is linear with respect to the increments of the components of the deviators of stress and strain; (3) the continuity condition is valid and, finally, (4) $dI_2 > 0$ is the critical loading criterion.

It follows from the first two conditions that

$$d\vartheta_{ij} = A_{ijke} dS_{ke} \tag{11}$$

where A_{ijke} depend on the stress deviator only. Assuming that

$$d\vartheta_{ij} = d\vartheta_{ij}^{\,y} + d\vartheta_{ij}^{\,p}$$

where $d\vartheta_{ij}^{\,y} = dS_{ij}/2G$ is the reversible elastic part, and $d\vartheta_{ij}^{\,p}$ is the non-reversible plastic part, we obtain for $dI_2 > 0$

$$d\vartheta_{ij}^{\,p} = C_{ijke} dS_{ke}$$

where G_{ijke}, as well as A_{ijke}, is a fourth rank tensor depending only on S_{ij}.

It follows from the continuity condition at neutral loading that $C_{ijke}dS_{ke} = 0$ when $dI_2 = 0$. Since $dI_2 = 0$ means that $S_{ke}dS_{ke} = 0$, a simultaneous inversion of two linear forms with respect to dS_{ke} leads us to

$$C_{ijke} = G_{ij} S_{ke}$$

and hence

$$d\vartheta_{ij}^{\,p} = \begin{cases} G_{ij} dI_2 & (dI_2 > 0) \\ 0 & (dI_2 \leqslant 0) \end{cases} \tag{12}$$

Tensor analysis permits to conclude that

$$G_{ij} = P(I_2, I_3^2) S_{ij} + Q(I_2, I_3^2) I_3 t_{ij} \tag{13}$$

is the most general form of G_{ij}.

But if we assume that I_3 has but little influence on the deformation, then we can consider G_{ij} a function of I_2 only, so that

$$d\vartheta_{ij}^{\,p} = \begin{cases} P(I_2) S_{ij} dI_2 & (dI_2 > 0) \\ 0 & (dI_2 \leqslant 0) \end{cases} \tag{14}$$

This is the simplest form of the law, as proposed by Laning.

Following the work of Handelman, Lin and Prager, the attention of researchers was on one hand directed toward experimental checking of the law (14) at complex loading, whereby most of the subsequent experiments confirmed that this law is closer to reality than the deformation theory. Theoretical research was continued, on the other hand, the purpose of which, essentially, was to broaden the differential law of plasticity for an arbitrary loading criterion. An increasingly important role in this research was assigned to the vectorial tensor representation.

Any tensor having n components, as any system of n numbers, can be represented in infinitely many ways in an n-dimensional vectorial space. If among the n components, $r < n$ are independent, then the corresponding vector is located in a subspace having r dimensions. In this manner, at any given point of the body, both the states of stress and strain can be represented. Taking advantage of the numerous possibilities of vectorial tensor representation, the correlation between the tensor and vector systems can be so determined, that not only the tensors are represented by vectors, but that there is also a vectorial representation of the most important tensorial operations. In our case of the stress-strain state, it is convenient to proceed in the following way.

We place the stress vector **P** with components p_i in correspondence with the stress deviator S_{ij}, and the strain vector Э with components $э_i$ with the strain deviator $\mathcal{Э}_{ij}$, in accordance with the law

$$\begin{matrix} p_1 = S_{11}, & p_2 = S_{22}, & p_3 = S_{33}, & p_4 = S_{13}, \ldots, & p_9 = S_{32} \\ э_1 = Э_{11}, & э_2 = Э_{22}, & э_3 = Э_{33}, & э_4 = Э_{13}, \ldots, & э_9 = Э_{32} \end{matrix} \quad (15)$$

Inasmuch as the stress and strain deviators have, in the most general case, only five independent components, the above-mentioned vectors **P** and Э are in reality in a five-dimensional subspace of the nine-dimensional space. For some theoretical investigations it is essential to consider just that five-dimensional vectorial space as a basic space, but then the relations between the vector and tensor components are not as simple as in (15).

Further in the text we will study the relation between the stress deviator and the plastic part of the strain deviator

$$Э_{ij}^{p} = Э_{ij} - Э_{ij}^{y} = Э_{ij} - \frac{S_{ij}}{2G}$$

and we will put the plastic strain vector $Э^p$ in correlation to the deviator $Э_{ij}^{p}$, which is easily determined on the basis of (15) as

$$\mathbf{Э}^p = \mathbf{Э} - \frac{\mathbf{P}}{2G}$$

When the dependence of $\mathbf{Э}^p$ on **P** for an arbitrary loading path is determined, the problem of the relation of Э and **P** is also solved.

In the process of deformation the tip of each vector describes a certain path. The paths described by the tips of the vectors **P**, Э and $Э^p$, we will call loading, strain, and plastic strain paths, respectively.

A special place among the loading paths is occupied by such a loading path, along which the stress deviator increases proportionally to one parameter. We have referred above to such loading as simple loading. In simple loading the trajectory of the stress vector (loading path) is a position [or radius] vector; the corresponding path of plastic deformation and of total deformation accordingly, will also be radial, whose direction coincides with the loading path and, independently of the loading direction,

$$Э = \frac{\mathbf{P}}{2G_s} \quad (16)$$

In fact, for simple loading from natural state, the Hencky-Nadai law is valid to a sufficient degree of accuracy, and it is easy to see that on the basis of (15) the vectorial representation of this law coincides with (16). Thus, the vectorial space is isotropic with respect to any simple loading. Or, in other words, the properties of the relation between stress and strain vectors for simple loading are invariant with respect to the rotation group in vectorial space.

Let us now consider some complex loading paths. For such paths, most likely, the same invariability is true, which now should be naturally complemented with the invariability with respect to the reflection of trajectories in all possible planes and directions. The hypothesis concerning the invariability of vectorial space was originally formulated by Il'iushin [14] and called by him the postulate of isotropy. It should be noted that Il'iushin formulated the postulate of isotropy as applied to the above mentioned five-dimensional vectorial space, where each coordinate is independent. But inasmuch as the space with which we are dealing (15) can be interpreted as a space in which such a five-dimensional space is imbedded, the isotropy postulate retains here the same sense: if some loading path is obtained from a given path (to which the given deformation path corresponds) through some operation of rotations and reflections (orthogonal transformation), then the same operation leads to the corresponding deformation path derived from the given deformation path. Or, in other words, the inner geometry of the deformation path is entirely determined by the inner geometry of the loading path, and vice versa.

The postulate of isotropy leads us to more specific forms of the general tenso-linear relation (5). Using this postulate, only five scalar functions of the invariants of the inner geometry of the loading path or deformation path (Il'iushin's five-term formula) can be unknown, as compared to an infinite number of scalar functions in relations (5). We note that if the isotropy postulate is valid for the stress-strain relation, it is also valid for the relation between the vectors of stress and plastic strain.

Let us return now to the question with which we are concerned regarding the connection between the loading criterion and the stress-strain relation.

It is known from numerous experiments with a sufficient degree of accuracy, that plastic deformations begin to appear in an initially isotropic material when the value of the second invariant of the stress deviator I_2 increases beyond a certain critical value

$$I_2 = \frac{1}{2} k^2 \tag{17}$$

In vectorial space this condition is represented by a sphere of radius k. Indeed, by definition

$$I_2 = \frac{1}{2} \Sigma S_{ij}{}^2 \tag{18}$$

and since according to (15) $\Sigma\, S^2_{ij} = \Sigma\, p_i^{\,2}$, hence

$$I_2 = \frac{1}{2}\Sigma p_i^2 = \frac{1}{2}|\mathbf{P}|^2 \tag{19}$$

Accordingly, condition (17) can be rewritten as

$$|\mathbf{P}| = k \tag{20}$$

Let the magnitude of the stress vector in the process of a subsequent change of stresses become somewhat larger than k. If the body were to be unloaded now (a homogenous state of stress is assumed), then the appearance of new plastic deformations in such a body which is plastically deformed, will not correspond to (17) any more. This will be some new closed surface in vectorial space which is cutomarily called loading surface (or flow surface †). Such a characteristic surface exists at each loading moment. It separates all elastic states, i.e. those states which can be reached from the given one without change of the plastic part of the deformation. Shape and size of that surface change with the change of stresses, if at the same time a change of the plastic part of the deformation takes place, and can be analytically represented as

$$f(S_{11},\ S_{22}, \ldots, \mathfrak{I}^p_{11}, \ldots, \mathfrak{I}^p_{13}) = \text{const} \tag{21}$$

Function f is usually called the loading function.

It is known from experiments, that a material can be elastically deformed from any state (elastic unloading). Hence it follows that in a continuous deformation process ($\mathfrak{I}^p$ changes continuously), the loading surface changes in such a manner that it passes all the time through the tip of the stress vector. The point on the loading surface which is touched by the tip of the stress vector in the plastic deformation process, we will call in the future the loading point.

For changes of $\mathfrak{I}^p$ at each given moment it is necessary to pass outside the loading surface constructed for the previous moment. Thus the loading surface is intimately connected with the loading criterion. If the loading surface is taken according to (21), then in view of the aforementioned, the loading criterion will be the condition

$$\frac{df}{dS_{ij}}\, dS_{ij} \geqslant 0 \tag{22}$$

when the equality sign corresponds to neutral loading. If $(\partial f/\partial S_{ij}) \times dS_{ij} < 0$, we have a case of unloading. In the Hencky-Nadai and Handelman-Lin-Prager laws the condition $dI_2 > 0$ was accepted as the loading criterion. The corresponding loading surface is a sphere of the radius $p = |P|$. Accordingly, the introduction of the criterion $dI_2 > 0$ presupposes that the elastic range is experiencing a simple isotropic expansion in the process of plastic deformation, which corresponds to uniform work-hardening of the material. This is, however, not true, or true only approximately. Generally speaking, the process of work-hardening is a directed process, and the work-hardening itself has to be anisotropic.

† The Mises sphere $|P| = k$ is the initial loading surface.

(Since we limit ourselves here to the review of small deformations only, the anisotropy which occurs in the plastic deformation process has to be explained by the influence of the state of stress. Such a form of anisotropy is called the anisotropy caused by stresses [12]).

The task of experimental investigation of such changes of the loading surface is rather complex, However, the mere concept of the loading surface, or loading function (which is the same), opens tremendous theoretical possibilities. It turned out that the loading surface is not only used for determination of whether the material is being loaded or unloaded at the given point, but is related (associated) with the plasticity law itself, so that a knowledge of the loading function permits to concretize substantially the stress-strain law. Furthermore, it became possible, using some rather general assumptions regarding the behavior of materials, to obtain some information with regard to the loading surface itself.

Going somewhat back, we will note that when we derived formula (14), the loading criterion was formulated in advance, i.e. the law of changes of the loading surface was determined without consideration of assumptions regarding stress-strain relations. It is interesting to see how the introduction of a different loading criterion will influence the final result, and to determine the relationship between the plasticity law and the loading criterion.

Results of investigations of this type given in Prager's article [15] can be summarized as follows. We will assume, just as we deduced the law (14), that: (1) the increments of the components of the strain deviator are completely determined by the stress deviator and its increment, (2) this relation is linear with respect to the increments, (3) the continuity condition is valid.

Inasmuch as we will not formulate the loading criterion in advance, and in order to obtain the most concrete relation between stresses and strains, we will require that the following supplementary conditions be satisfied: (4) when given forces are acting on the surface of the body, small increments of plastic deformation within the body are uniquely determined by the given small increments of surface and body forces (condition of uniqueness in the small), (5) loading surface at loading point has only one normal (condition of regularity of the loading surface) and (6) the loading surface changes continuously during the plastic deformation process (continuity condition of the change of the loading surface).

Let us consider some given state of stress characterized by stress vector **P**. We will consider only the case when the tip of the stress vector is touching the loading surface. Otherwise we would find ourselves in the elastic range, the laws for which have already been determined. If the tip of the stress vector is touching the loading surface, then an infinitely small change of the state of stress $d\mathbf{P}$ will either cause loading, or neutral loading or unloading, depending on whether $d\mathbf{P}$ is directed inside, tangentially to or outside the loading surface. To every vector $d\mathbf{P}$ directed outside the loading surface, there corresponds uniquely a vector $d\mathcal{E}^p$. The requirement of the continuity requires that as $d\mathcal{E}^p \to 0$ the vector $d\mathbf{P}$ is approaching the tangential direction to the loading surface. A vector $d\mathbf{P}$ directed outside the loading surface can be represented as a sum $d\mathbf{P} = d\mathbf{P}_t + d\mathbf{P}_n$, where $d\mathbf{P}_t$ and $d\mathbf{P}_n$ are vectors along the tangent and along the normal to the loading surface, at loading point $\mathbf{P}$. But according to the continuity condition, the component $d\mathbf{P}_t$ does not affect a change of $\mathcal{E}^p$, hence a change of the plastic deformation takes place only on account of $d\mathbf{P}_n$, and the direction of $d\mathcal{E}^p$ does not depend on the direction of $d\mathbf{P}$. It follows, that with each point $\mathbf{P}$ of the loading surface there is associated a corresponding unit vector n in such a way, that the increment of plastic deformation is effected by an infinitely small increment of the stress vector $d\mathbf{P}$ in the direction $\mathbf{n}$. Let us now investigate the relation between the unit vector $\mathbf{n}$ and the loading surface.

We will turn our attention to a body of work-hardening material with given surface and body forces $\mathbf{T}$ and $\mathbf{F}$. The theorem of virtual work states that

$$\int_s \mathbf{T}\cdot\mathbf{u}\,ds + \int_v \mathbf{F}\cdot\mathbf{u}\cdot dv = \int_v \sigma_{ij}\varepsilon_{ij}dv \tag{23}$$

Here $\mathbf{T}$, $\mathbf{F}$, σ_{ij} correspond to equilibrium, and $\mathbf{u}$, ϵ_{ij} are possible. Assume that under the given increments of external forces, two states $d\sigma_{ij}^{(1)}$, $d\epsilon_{ij}^{(1)}$, $d\sigma_{ij}^{(2)}$, $d\epsilon_{ij}^{(2)}$ are possible within the body. The differences

$$\Delta d\,\sigma_{ij} = d\sigma_{ij}^{(1)} - d\sigma_{ij}^{(2)}, \qquad \Delta d\varepsilon_{ij} = d\varepsilon_{ij}^{(1)} - d\varepsilon_{ij}^{(2)}$$

can be substituted into (23), even though $\Delta\, d\sigma_{ij}$ may not necessarily correspond to $\Delta\, d\epsilon_{ij}$. Hereby the left-hand side in (23) becomes zero, and we obtain

$$\int_v \Delta d\sigma_{ij}\Delta d\varepsilon_{ij}dv = 0 \tag{24}$$

Thus, to satisfy the condition of uniqueness the requiremnt that the integrand be non-negative, is sufficient. In fact, from (24) it will follow then, that the solutions coincide. If we consider separately the elastic part in the integrand, which is always positive, then it is sufficient to require that $\Delta\, d\sigma_{ij}\Delta\, d\epsilon_{ij}^{p} \geqslant 0$. Since ϵ_{ij}^{p} is a deviator, it follows that $\Delta\, dS_{ij}\Delta\, d\mathcal{E}_{ij}^{p} \geqslant 0$ or, in vectorial form

$$\Delta d\mathbf{P}\cdot\Delta d\mathcal{E}^p \geqslant 0 \tag{25}$$

Let us now consider three possible cases: (a) both solutions correspond to loading, (b) one is unloading, the other loading, (c) both solutions result in unloading.

In the last case $\Delta d\partial^p = 0$, and, accordingly, the integrand in (24) is positive. Taking advantage of the fact, that in the other two cases the sequence in which the solutions are taken, does not influence the sign of $\Delta d\mathbf{P}\, \Delta d\partial^p$, we can select them in such an order that $\Delta\, d\mathbf{P}$ be directed toward the outward normal. In case (a), in view of the presupposed linearity of the relations of increments of stress and strain $\Delta d\partial^p$ there is a solution which corresponds to $\Delta\, d\mathbf{P}$, and, in consequence, is directed along **n**. Thus in that case the inequality (25) expresses the condition that the scalar product of the vector **n** by any vector in the direction of the outward normal to the loading surface be non-negative. Consequently, vector **n** can only be the unit vector of the outward normal to the loading surface at point **P**. Accepting this definition for **n**, it is easy to see that the expression for $\Delta d\mathbf{P}\, \Delta d\partial_p$ becomes non-negative in case (b) as well.

Thus we arrived at the conclusion that the vector of increments of deformation is directed along the outward normal to the loading surface at point **P**.

It remains now only to determine the value of the scalar proportionality coefficient in the law $d\partial^p = dk\mathbf{n}$, which in tensorial representation may be written down as:

$$d\partial_{ij}{}^p = dk\,(\partial f \,/\, \partial S_{ij}) \tag{26}$$

To determine dk we will use the continuity condition of the loading surface change, in view of which, if at the given state

$$f\,(S_{11}, \ldots, \partial_{11}{}^p, \ldots) = \text{const},$$

then for a neighboring state

$$f\,(S_{11} + dS_{11}, \ldots, \partial_{11}{}^p + d\partial_{11}{}^p, \ldots) = \text{const}$$

and, therefore

$$f\,(S_{11} + dS, \ldots \partial_{11}{}^p + d\partial_{11}{}^p, \ldots) - f\,(S_{11}, \ldots, \partial_{11}{}^p) = 0$$

Hence,

$$(\partial f \,/\, \partial S_{ij})\, dS_{ij} + (\partial f \,/\, \partial \partial_{ij}{}^p)\, d\partial_{ij}{}^p = 0 \tag{27}$$

Introducing (26), we arrive at a formula for dk

$$dk = -\left(\frac{\partial f}{\partial S_{ij}}\, dS_{ij}\right) \Big/ \left(\frac{\partial f}{\partial S_{ij}}\, \frac{\partial f}{\partial \partial_{ij}{}^p}\right) \tag{28}$$

and thus

$$d\partial_{ij}{}^p = F\, \frac{\partial f}{\partial S_{ij}} \left(\frac{\partial f}{\partial S_{ij}}\, dS_{ij}\right), \qquad \frac{\partial f}{\partial S_{ij}}\, dS_{ij} \geqslant 0 \tag{29}$$

where

$$F^{-1} = -\frac{\partial f}{\partial \partial_{ij}}\, \frac{\partial f}{\partial \partial_{ij}{}^p} \tag{30}$$

In deriving this law we have assumed an obvious dependence on plastic deformation. In the case when f obviously does not depend on plastic deformations, it is impossible to determine the coefficient through f. Taking advantage of the indeterminacy of dk in that case, we can again represent the result in form (29). Of course, formula (30) is not applicable in that event, and F has to be determined independently of f.

The problem of investigation of various analytical expressions of the loading function has gained considerable importance. Research in that field has been conducted by Edelman and Drucker [16].

Any loading function has to satisfy the law of simple loading. That means that the relation (29) for the given function has to correspond satisfactorily with the experimental results of simple loading. Naturally, the results will be more complete when the third invariant of the stress deviator is considered, rather than when this consideration is omitted. The most simple example of the loading function is $f = I_2$, or $f = f(I_2)$. If, in addition to this, we assume that $F = F(I_2)$, Laning's law will follow from formula (29).

The function $f = f(I_2, I_3^2)$ corresponds better to simple loading data; however, as well as the preceding one, it does not take into consideration the anisotropy due to stresses. By introducing the plastic deformations into the loading function, we can take into consideration the last effect as well as the Bauschinger effect, as for example

$$f = \Phi(I_2) - mS_{ij}\vartheta_{ij}^p,$$

or, if in addition to that the third invariant is also considered,

$$f = \Phi(I_2, I_3^2) - mS_{ij}\vartheta_{ij}^p$$

It becomes apparent here that certain limitations have to be imposed on Φ.

Thus it is quite obvious that a taking into account of the Bauschinger effect and of the anisotropy which takes place during the loading process, leads to very complex plasticity laws which are scarcely applicable for practical use. Laning's law is therefore the most frequently used plasticity law, even though it is also the most approximate law of all.

We have examined here in detail the logical considerations which result in the law (29), sometimes called the Hodge-Prager law. A change of the original assumptions results in a change of (29) and thus some other formulations of the plasticity law can be obtained. Assuming that deformations depend mainly on the loading history, Cunningham, Thomsen and Dorn [17] arrived at the law

$$\frac{S_{ij}}{\sqrt{I_2}} = \frac{d\vartheta_{ij}^p / dt}{\sqrt{K_2}}, \qquad I_2 = F\left(\int \sqrt{K_2}\,dt\right) \tag{31}$$

where K_2 is the second invariant of the deformation velocity [strain rate] deviator. However, Prager [12] showed that this in essence coincides with (14). If F is a monotonous function we can write down

$$\int \sqrt{K_2}\,dt = \Phi(I_2)$$

Upon differentiation we will obtain $\sqrt{K_2} = \phi\,(I_2)(dI_2/dt)$, so that introducing this into law (31), we will again arrive at Laning's formula, where we only have to put

$$P(I_2) = \varphi(I_2)/\sqrt{I_2}$$

The most important factor in the change of the system of assumptions was the introduction of the work-hardening postulate, which was contained in the finalized version in Drucker's paper [18].

Below follows an interpretation of uniaxial work-hardening properties.

In work-hardening materials the increments of strain and the corresponding increments of stress are such that their product is positive. If, in addition to the process of application of supplementary increments of stress, the process of their removal is considered also, then it can be said that in the indicated cycle the work of stress increments in strain increments is positive. The postulate of work-hardening is precisely a generalization of this property of the curve from a uniaxial test to the polyaxial case. It appears that the acceptance of that generalization puts some very rigid requirements on the plasticity law. As it was shown by Drucker not long ago, violation of the fundamental postulate of work-hardening can lead to an indeterminacy of the solution [19], and thus a fulfilment of it for real materials in all probability is necessary.

Let vector $\mathbf{P}^*$ represent a state of stress at a given point of an elastic-plastic body, and let the tip of this vector be either within or on the loading surface. Let us further assume that additional external forces create at the given point of the body additional stresses, which displace the tip of the stress vector from within the loading surface to some point $\mathbf{P}$ on this surface. Only elastic deformations are produced in this process, and since elastic deformations are reversible, the state of strain at point $\mathbf{P}$ will not depend on the path of the stress vector from point $\mathbf{P}^*$ to $\mathbf{P}$. Further, supplementary external forces bring the stress vector outside the loading surface up to the value $\mathbf{P} + d\mathbf{P}$, so that small increments of plastic deformation occur. Then the supplementary external forces are removed just as slowly as they were applied, and the stress vector $\mathbf{P}$ returns to the original condition $\mathbf{P}^*$ along some path (we are considering a homogenous state of stress). Since elastic work is reversible, the total work for the complete cycle of application and removal of supplementary forces will be

$$dW = (\mathbf{P} - \mathbf{P}^*)\cdot d\,Э^p + d\,\mathbf{P}\cdot d\,Э^p$$

According to the work-hardening postulate, this work will be positive. Since, in a particular case, we can take $\mathbf{P}^*$ for $\mathbf{P}$, the work-hardening postulate yields

$$d\mathbf{P}\cdot d\,\mathfrak{Э}^p > 0 \tag{32}$$

If $\mathbf{P}-\mathbf{P}^* \neq 0$, then this difference can be made arbitrarily larger than $d\mathbf{P}$, and therefore

$$(\mathbf{P}-\mathbf{P}^*)\cdot d\,\mathfrak{Э}^p \geqslant 0 \tag{33}$$

Thus, on the basis of the work-hardening postulate, the vectors $\mathbf{P}-\mathbf{P}^*$ and $d\,Э^p$ have to form an acute angle, and accordingly all possible vectors $\mathbf{P}-\mathbf{P}^*$ must be located on one side of the plane perpendicular to the vector $d\,Э^p$, and this should be true of all the $\mathbf{P}$'s on the loading surface. It follows that the loading surface is non-concave throughout. If it is true that the loading surface at a loading point has only one normal, it is easy to see from (32) that $d\,Э^p$ is directed along the normal to the loading surface at point $\mathbf{P}$. It should be noted that $d\,Э^p$ cannot be directed inward to the loading surface, since $d\mathbf{P}\cdot d\,Э^p > 0$, and $d\mathbf{P}$ is directed outward.

Thus, only from accepting the postulate of work-hardening and from the condition of regularity of the stress point it follows that

$$dЭ_{ij} = dk\,(\partial f / \partial S_{ij})$$

The condition

$$\frac{\partial f}{\partial S_{ij}}\, dS_{ij} > 0 \tag{34}$$

remains, as previously, the loading criterion

Inasmuch as dk is undetermined, it is justified to say thatt

$$dЭ_{ij}{}^p = F\,\frac{\partial f}{\partial S_{ij}}\left(\frac{\partial f}{\partial S_{ij}}\, dS_{ij}\right) \tag{35}$$

where F may depend on stress, strain and deformation history. This function, in particular, may also depend on dS_{ij}, but then it has to be homogeneous of zero order with respect to dS_{ij}, since time effects in plasticity are excluded. For example

$$F = g\left[1 + \frac{(dS_{ij}\, dS_{jk}\, dS_{ke})^2}{(dS_{mn}\, dS_{mn})^3}\right]$$

where g does not depend anymore on dS_{ij}. The function F does not have to depend on dS_{ij} at all, if we require differential linearity of the relation between stress and strain.

We have considered various forms of differential stress-strain laws or laws of flow, which express the increment or differential of strain through some stress operator. In some applications, such as in problems of stability beyond the elasticity limit, those relations have to be solved with respect to the increments of stress. Such inversions of the laws of flow are sometimes connected with considerable difficulties, hence the idea of direct establishment of flow laws in terms of dependence of stress increments on strain, strain increments, etc., is quite justified.

Such a theory of flow in terms of stresses was developed by Trifan [20] and was based on the following suppositions: (1) The increments of stress are completely determined by strain and increments of strain, (2) this relation is assumed to be differentially linear, (3) the condition of continuity holds, and (4) the condition $dq > 0$ is accepted as the loading criterion, where q is a function of the invariants E_2 and E_3 of the strain deviator. For simplicity purposes, we limit ourselves to the consideration of only one specific case, namely when q is a function of E_2 only and, accordingly, it can be assumed that $dq = \partial_{ij}\, d\partial_{ij}$. (Taking into account the third strain deviator leads to a tensorially non-linear theory, which will not be considered here).

If $dq = \partial_{ij}\, d\partial_{ij}$, then it is easy to see that the system of the above-mentioned four suppositions is simply a reversal of the system used by Handelman, Lin and Prager. Therefore, the derivation of the expression for dS_{ij} will be almost a literal repetition of the deductions made by those three authors. As a result it is easy to obtain

$$dS_{ij} = 2G\, d\,\partial_{ij} - k_{ij}\, dq \tag{36}$$

where

$$k_{ij} = P(E_2)\, \partial_{ij} \tag{37}$$

Flow theories analogous to the one just presented have received little popularity, and as far as we know no attempt has been made to generalize them with respect to an arbitrary loading criterion.

In the entire material which was discussed previously, the regularity of the loading function surface at the point of loading was assumed. However, already the application of the criterion of maximum shear stress offers an example of a loading surface with some singular points. It is true, though, that in this case the singular points are stationary and may not correspond with the loading point. It can be expected, however, that it is just the loading point that at all times is singular. It is possible that such a supposition occurred for the first time in the explanation of the torsional loss of stability; there is no doubt, however, that the main role in establishing the concept of the singular loading point was played by investigations of plasticity properties, based on considerations of the microstructural mechanisms of plastic slip.

Essential in such investigations is the consideration of the crystalline structure of the material. The real body is visualized as an aggregate of a large number of arbitrarily oriented monocrystals. Consequently, the plastic properties of the body may be assumed as statistical means of plasticity properties of individual crystals and also of the properties of their interaction. The last two groups of properties we will call microplasticity properties. Modern plasticity theories based on such an approach are frequently called physical theories, which is a hardly justifiable designation. A truly physical theory we would consider a theory which, based on microplastic properties, would yield all the macrocharacteristics of the material. But contemporary schools of thought in that direction follow a so to speak "semi-inversed" way to describe the combined loading process. Instead of giving completely the microplastic properties, they are prescribed with some amount of indeterminacy, and the latter is eliminated later only in an average way with the aid of reference to the macroexperiment. This leads to only a conditional acceptance of the term "physical" for the theories mentioned below, quite aside from the fact that the microplastic properties with which those theories deal are rather primitive.

It is known from experiments [21] that plastic deformation in a monocrystal is essentially determined by shearing along definite planes, called glide [or slip] planes, and the directions in them, called directions of slip. The slip plane, together with a direction of slip on it, is known as a slip system. The magnitude and orientation of slip systems is determined by the form of the crystal lattice of the material.

It should be stressed that in order to determine the orientation of a slip system, i.e. of a system consisting of a plane and a direction, three independent parameters have to be given.

Since it has been established that the pressure normal to the slip plane has hardly any influence on plastic deformation of the crystal, its plastic properties are determined by the relation of shear stress components in slip systems to the plastic shear strains in those systems.

Various theories differ in the manner in which they postulate the plasticity properties in the slip systems (in the following we will designate the summation of such properties by the letter A), and the way they determine the properties on crystal boundaries (B properties). The material is considered to be quasi-isotropic, and as a result of that any orientation of slip systems is presumed to be equally probable in volume, containing a sufficient amount of monocrystals (for example, the experimental tubular specimen).

The first theory proposed by that school of thought for description of plasticity properties at combined loading, was the so-called slip theory of Batdorf and Budiansky [22]. Following is the essence of the basic assumptions of this theory:

A. It is assumed that each monocrystal has only one slip system. The plastic shear γ^p in this system depends on the maximum component of shear stress assumed during the entire loading history and takes place only if τ becomes larger than some limiting value τ_L.

B. The state of stress in each crystal is the same and coincides with the state of stress of the entire aggregate.

Let the parameters, which determine the orientation of some slip system with respect to the fixed axes, be α, β, γ. The component τ of the shear stress active in this system, can be expressed through stresses in the fixed axes x, y, z with the aid of α, β, γ. The plastic shear in this slip system will be by definition

$$\gamma^p = F(\tau), \qquad \tau > \tau_L$$

where F is a still unknown function. Through γ^p and α, β, γ we can determine the plastic strains ϑ_{ij}^p, which occur along axes x, y, z as a result of plastic deformation in systems with the given orientation

$$\vartheta_{ij}^p = F(\tau)\,\varphi_{ij}(\alpha, \beta, \gamma)$$

where ϕ_{ij} is a determined function of its variables. Since any orientation of slip systems is equally probable, we can, by means of averaging, obtain ϑ_{ij}^p for macrodeformation

$$\vartheta_{ij}^p = \iiint F(\tau)\,\varphi_{ij}(\alpha, \beta, \gamma)\,d\alpha\,d\beta\,d\gamma \tag{38}$$

Actually, we performed a summation of all possible orientations, but in view of the indeterminacy of the function F we can consider this an averaging.†

The function F, or the characteristic function, can be determined, for example, from an experiment on uniaxial tension by the method of series expansion and subsequent numerical integration. A specific form of dependence of F from τ/τ_L and, accordingly, dependence of plastic shearing from shear stress in the unique slip system of a monocrystal, is shown in Fig. 1.

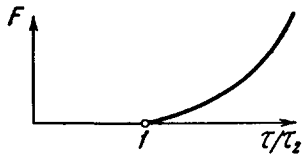

Fig. 1.

Thus, to obtain a logical inference of the macro-connection "stress-strain" from premises A and B, which would not contradict the experiment on uniaxial loading, it is necessary to assume that the monocrystal possesses work-hardening in its unique slip system.

† It seems to us that the method of deduction used here, which differs from the one employed by Batdorf and Budiansky [22], emphasizes stronger the "physics" (in the above-named limited sense) of the slip theory.

Let us bring into evidence the main qualitative results that follow from the slip theory. Let a simple loading beyond the limit of elasticity take place at a given point of the body. If the material was initially isotropic, then the loading surface has to be symmetrical with respect to the loading direction. It follows that if there is no singularity at the loading point, then any orthogonal loading that follows the simple loading (at a right angle to the previous simple loading) will be neutral at the initial moment. At this initial moment the orthogonal loading is characterized by $|\mathbf{P}| = p =$ const., i.e. the stress intensity is constant. In view of the continuity condition we are justified to say that if an increment of plastic deformation is obtained at the initial moment of orthogonal loading, then there has to be a conical singularity at the loading point.

The slip theory precisely predicts an increment of plastic deformation for orthogonal loading. Indeed, at simple loading an unequal work-hardening in different crystals takes place. Moreover, some crystals may not harden at all. If orthogonal loading takes place thereafter, then in view of redistribution of stresses a plastic deformation in some crystals which did not harden yet, or did not harden sufficiently, will set in. This will result in an increment of the overall plastic deformation.

A particular case of orthogonal loading is the pure shear which follows simple tension-compression. In tests on tubular specimens this is realized by torsion at constant axial stress. As we have observed, increments of plastic deformation even at the initial moment have to occur. It is interesting to determine the direction of the vector of increments of plastic strain in this process. A partial answer to this question is given by the value of the so-called instantaneous shear modulus

$$\left.\frac{d\tau}{d\gamma}\right|_{d\sigma = 0} = G_i$$

where σ and τ are tension-compression and shear stresses, respectively and γ is the total shear strain.

If $G = G_i$, i.e. the elastic shear modulus equals the instantaneous plastic modulus, then the vector of increments of plastic deformations at orthogonal loading is either a zero vector (no singularity at loading point), or has the direction of initial simple loading. If, however, G_i differs from G, then the conical singularity necessarily exists (if the continuity condition is valid) and the vector of increments of plastic deformations makes a finite angle with the initial simple loading direction.

Direct calculations made by Cicala [23] on the basis of the slip theory, resulted in the following values for G_i :

$$G_i = \frac{G}{1 + \frac{3G}{2}\left(\frac{1}{E_s} - \frac{1}{E}\right)}$$

Authors of this theory have jointly with other researchers conducted an experimental check of the slip theory [24], and established some of its advantages compared to the flow and deformation theories. However, in the initial tests on experimental determination of the instantaneous shear modulus G_i, it was not possible to discover a sizeable deviation from the elastic value. A deviation (smaller though, than suggested by the slip theory) was only discovered somewhat later, and it is quite possible that lack of success of the initial tests in determining G_i was due to the fact that the torsion of the tested specimens occurred at very small (elastic range) axial plastic deformation.†

At any rate, it can be safely stated that the predictions of the slip theory, as to the value of the instantaneous shear modulus, could stand some improvement. As for the reason of the inadequacy of this theory, it is contained in its basic assumptions, First (point A), the supposition of the uniqueness of the slip system in a monocrystal is a great simplification of the actual state of affairs. Secondly and mainly, the supposition expressed in point B as to the random orientation of crystals, in essence contains the possibility of fracture on crystal boundaries and is chiefly responsible for the inadequacy of the theory.

Even though the deficiencies of the basic assumptions in the final law relating stresses and strains are to some extent improved by the determination of the characteristic function from the macroexperiment, the slip theory still remains extremely simplified.

In 1954 Lin proposed a new version [25], whose aim it was to improve some of the deficiencies of the slip theory.

Lin's basic assumptions are reducible to the following.

A. Several slip systems exist in a monocrystal. The active ones (i.e. such systems in which a plastic shear takes place at the given moment) are systems which correspond to the minimum work at the given deformation of the crystal. The shear stress components in such active systems are equal to each other, and their magnitude depends on the sum of slips in the given crystal.

B. The deformation of all crystals is alike and coincides with the deformation of the aggregate as a whole. The state of stress of the aggregate is obtained as an average of states of stress of component crystals.

The assumptions A and B are still too complex for the derivation of a

† This supposition was first expressed by A.M. Zhukov and Iu. N. Rabotnov in their paper [30].

stress-strain relation. Therefore, the author introduces additional suppositions of a simplifying nature. Inasmuch as it is not our objective to obtain a mathematical formulation, we will not consider Lin's simplified theory. We will only state that in our opinion, contrary to Lin's assertion, the equality of G_i and of the elastic shear modulus does not follow from the simplified version of the theory, whereas it is probably true for the initial suppositions A and B.

Some basic qualitative conclusions from the suppositions A and B may be easily obtained from the model of a "plane" polycrystalline body. We have in mind an aggregate of monocrystals, having the following properties: in each monocrystal of the aggregate, the shears take place in one plane only, which is a common plane for all of the crystals. Otherwise, the orientation of the crystals is arbitrary. Of course, in order to fully utilize the simplifying properties of a "plane" body, we have to apply the external loading, too, in a plane which is characteristic for this body.

Let us assume that there are three possible directions of slip in the characteristic plane of each crystal which form 60° angles to each other. (This is characteristic for slip planes of metals with a face-centered cubic lattice, typical for aluminum and its alloys). The active directions will be only the two that are closest to the direction of the resulting shear. Inasmuch as the shear stress components along these are always equal, the resulting shear stress is always along the bisecting line of the angle, which is defined by those two directions. Since the direction of the resulting shear strain also lies within this angle, the maximum difference between it and the resulting shear stress can amount to 30°. In simple loading the direction of the resulting shear strain γ_0 is constant, and according to B equal for all crystals. In view of that, the active slip directions are constant in each crystal. Let us now assume that an increment of stress has been produced, which is other than simple. This will result in an additional shear strain in each crystal (Fig.2). If the direction of such an additional shear in some crystal does not go beyond the limits of the angle formed by the initial directions of shear, there will be no change of the active slip directions and the direction of the resulting shear stress in this crystal will remain unchanged. However small the angle α made by the vectors $\Delta\gamma$ and γ_0 might be, there will always be a crystal on the characteristic plane of the given crystal, whose active slip directions will undergo changes, (γ_1 to γ_3. This will take place in such crystals where the direction γ_2 is closer to γ_0 than $\Delta\gamma$). In these crystals a directional change of the resulting shear stress will take place, and consequently, an increment of the shear stress $\Delta\tau$ will also take place. For the whole body these increments will result in some increment of the total (average) stress. It is not hard to notice that the relation between this increment and the angle is continuous and that $\Delta\sigma_{ij\ \text{total}} = 0$ corresponds to $\alpha = 0$.

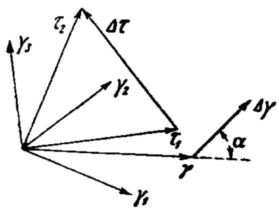

Fig. 2.

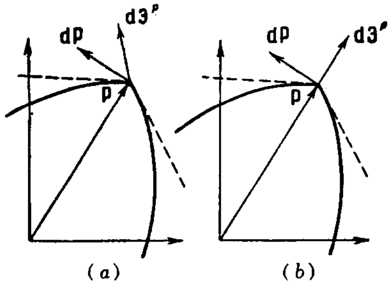

Fig. 3.

(The process of averaging is to be kept in mind; for example, when α is small, the amount of crystals in which a change of the active slip directions will take place, is also small, accordingly $\Delta\sigma_{ij\ \text{total}}$ will be small, too). Hence it can be asserted that during any sharp change of stresses (broken loading curve) in the considered body, the deformation curve will be smooth. That means that the instantaneous shear modulus will equal to its elastic value.

It is obvious from Fig.2. that without interrupting [stopping] plastic deformation in a crystal of average orientation, $\Delta\tau$ may have a component in a direction which is opposite to the direction of the initial simple deformation. This indicates the presence of a conical (angular) singularity of the loading surface (for a "plane" body of the loading curve) at the loading point.

We have considered the applications of the basic stipulations A and B to a plane scheme. The same conditions, only in a more complex form, will be true for a three-dimensional model.

Comparing the conclusions of the two theories considered, we might say that both the slip theory and Lin's theory predict the occurrence of a conical singularity at the loading point, but differ insofar as the slip theory predicts a breaking of the plastic deformation path at orthogonal loading, and Lin's theory does not. In Fig.3, *a* corresponds to slip theory and *b* to Lin's theory.

Finally, it should be mentioned that Lin's simplifying propositions reduce essentially to the assumption that every plane in a real material has the properties of the above-mentioned "plane" body. The behavior of such planes is assumed to be independent, and the total plastic deformation of the aggregate is obtained by summation of the plastic deformations on all such planes. In the case of orthogonal loading, besides the effects in a "plane" body, which we considered previously, the effect of inclusion of plastic behavior of non-hardened or not sufficiently hardened planes must take place, which, in principle, is similar to the slip theory. Hence, it is impossible to expect that the equation $G_i = G$ holds, as was pointed out above.

A theory recently proposed by Malmeister [26], is rather close to the two plasticity theories described above.

The following is the essence of the basic propositions of the Malmeister theory. It is assumed that plastic deformation in the neighborhood of the given point of the body is determined by shears in all possible planes. In every such plane the plastic shear depends on the shear stress on this plane only. The behavior in each plane is independent.

It is not hard to see that the mathematical formulation of Malmeister's theory will differ from the slip theory formulation only insofar as one integration will be missing (integration along the angle on each plane). Naturally, in relation to this, the characteristic function will be different from that in the slip theory. The qualitative result of that change will be such that the conicity of the loading point, as well as the breaking angle of the plastic deformation path at orthogonal loading, will be smaller than in the slip theory. Allthis is a result of the hardening process, which takes place uniformly in all directions on each plane, contrary to the slip theory. For an arbitrary change of shear stress on a given plane, plastic deformations do not take place with the decrease of its value. Thus, at orthpgonal loading, increments of plastic deformation occur only on account of the inclusion of the plastic behavior of non-hardened or not sufficiently hardened surfaces (their number is ∞^2), and not slip systems (∞^3), as in the slip theory. In connection with this, the total increment of strain will be smaller than in the slip theory, and a smoothing of all effects will take place.

Hence, from the standpoint of any of the three plasticity theories mentioned above, the loading surface in the process of plastic deformation changes so that the loading point in it is conical. From the mathematical formulation standpoint, their joint characteristic is that the relation "stress-strain" contains multiple integrals and involves considerable mathematical difficulties even in the determination of the dependence of deformation on loading for homogenous stress conditions. The concept of the conical loading surface enjoys a considerable popularity, and the natural tendency becomes apparent to evolve such a plasticity theory, which, along with greater simplicity, would incorporate this main feature of the considered theories.

The initial step in this direction was taken by Koiter [27] and resulted simply in an expansion of the Hodge-Prager relations by way of introducing numerous loading functions. This school of thought can be characterized as a "reconciliation" of the gradient principle (the plastic strain increment vector is a gradient of the loading function for a portion of the loading surface with a continuous tangential plane) and the concept of the conical point on the loading surface.

As previously stated, the loading function in the stress space determines the loading surface. In the case when the loading surface has some singularities it can be represented as the envelope of some regular surfaces. Each such surface determines some regular loading function f_α, and thus, in accordance with the Hodge-Prager law, determines the increment of the plastic deformation vector with the accuracy to a scalar factor. If we were to assume the independence of each of the above-mentioned loading functions, the total plastic deformation could be described as a sum of plastic strain increments determined by each loading function

$$\begin{gathered} d\Im_{ij}{}^{p} = \sum_{\alpha=1}^{n} G_\alpha \frac{\partial f_\alpha}{\partial S_{ij}}\left(\frac{\partial f_\alpha}{\partial S_{ij}} dS_{ij}\right) \\ G_\alpha \geqslant 0 \text{ if } \frac{\partial f_\alpha}{\partial S_{ij}} dS_{ij} \geqslant 0, \quad G_\alpha = 0 \text{ if } \frac{\partial f_\alpha}{\partial S_{ij}} dS_{ij} \leqslant 0 \end{gathered} \tag{39}$$

Koiter showed that the slip theory is a special case of the above-mentioned concept and corresponds to an infinite number of plane loading surfaces.

Plane loading surfaces are the most simple elements of such theory, and therefore future investigations of the relation proposed by Koiter were conducted on the basis of plane loading surfaces. Saunders' [28] paper is one of such investigations, where it is assumed that separate plane loading surfaces are displaced parallel to each other in the plastic deformation process. In this case each loading function is a linear function of stresses and the Hodge-Prager relations may be partially integrated for each one of them. (It is shown in the paper that conversely as well, unique integrated stress-strain relations will be those for which f = const. represents a plane in stress space). Saunders' implications clearly permit to construct a loading surface for any given point of any arbitrary loading path. Clearly, only such plane loading surfaces, which have one common point with the stress vector, are displaced in the plastic deformation process. In view of the independence of their actions, each loading plane can move in one direction only, opposite to the origin of the coordinates. The supplementary supposition on the parallelism of displacement of individual loading planes gives immediately the method for the construction of the loading surface, as is seen from Fig. 4 (plane case). This figure shows two versions for the construction of the loading surface: (a) if the initial loading surface (indicated by shading) is a polyhedron; (b) if the initial loading surface is smooth. In the last case the construction method is the envelope method, or the method of external tangents. Insofar as the total plastic deformation is determined by the sum of plastic deformations, which correspond to individual loading planes, it stands to reason that it will be equal for all loading paths which determine the same loading surface.

It is easy to see that if the initial loading surface is smooth and closed, such as for example, the Mises surface, then for the determination of the initial and of all subsequent loading surfaces an infinite number of plane surfaces has to be considered. From the set assumptions it follows then that a conical singularity appears at the loading point during a continuous plastic deformation.

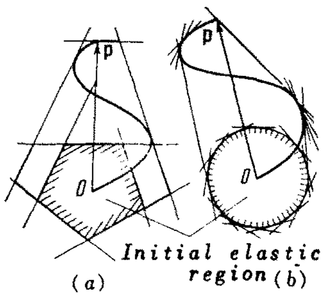

Fig. 4.

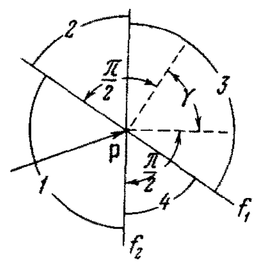

Fig. 5.

Below we will describe the results for a plane loading path when the plane surfaces can be represented as straight lines.

First, only two loading surfaces f_1 and f_2 are relevant at loading point (Fig.5). Secondly, the direction of the plastic strain increment vector depends on the direction $d\mathbf{P}$ and on the normals to the surfaces f_1 and f_2 in the following manner. If $d\mathbf{P}$ is directed toward the elastic region 1, then the plastic strain increment equals zero. If $d\mathbf{P}$ is directed toward region 3, then the plastic strain increment vector $dЭ^p$ can be only within the angle γ. If $d\mathbf{P}$ falls into the regions 2 or 4, then $dЭ^p$ coincides with the normal to f_1 or f_2.

It is also important to clarify the problem of differential dependence in Koiter's relations.

The most widespread plasticity laws (Hencky-Nadai, Laning) are differentially linear †, if, as it is usually the case with the majority of questions of practical importance, the unloading process is excluded. Thus, if we consider the continuous process of plastic deformation (active process), then the above-mentioned theories have all the remarkable properties of differentially linear relations. Contrary to that, the differential relations of Koiter, even in an active process, cannot be considered differentially linear in the full sense of the word. The truth of the matter is, that for many loading functions the presence of a "partial plasticity" region, such as regions 2 and 4 in Fig.5, is

† Linear with respect to stress and strain increments.

characteristic. These regions play the same role in the violation of the superposition principle, as do elastic regions in usual plasticity laws. But inasmuch as the existence of such regions is determined by the loading condition, it may be said that the differential non-linearity of Koiter's law is inherent in the loading conditions.

In this connection we will mention the plasticity theory recently proposed by Warner and Handelman [29]. Even though this theory in its final formulation does not essentially differ from Koiter's theory, it claims more generality and independence from the Hodge-Prager relations by its development of more general implications from a lesser amount of assumptions. However, the deductions themselves of Warner and Handelman relations seem questionable to us, since the authors use essentially the properties of differential linearity of the relation, which are subsequently violated by the accepted loading condition. Differential linearity of the relation "stress-strain" is obtained as a consequence of a series of assumptions regarding independence of plastic behavior from time and continuity of partial derivatives of strain rates as a function of stress velocities over the entire defined range. But as soon as a loading criterion, which accounts for the appearance of various plasticity regions is introduced, the second of the initial assumptions is violated.

The above-mentioned considerations regarding the behavior of a loading surface and, in particular, the existence of a conical point on the loading surface, require direct experimental investigation. As mentioned previously, initial attempts to determine the instantaneous shear modulus resulted in the equality $G_i = G$. This result appeared to be rather peculiar, especially in connection with the phenomena of torsional loss of stability. Therefore experimental attempts continue to prove that $G_i = G$. In 1954 two papers on the subject were published, one by Zhukov and Rabotnov [30] and the other by Naghdi and Rowley [31]. The first paper explores the behavior of tubular steel specimens, the second that of aluminum alloys. Both papers establish that the instantaneous shear modulus depends on the magnitude of tensile stress and may be considerably smaller than the elastic shear modulus. The latter fact was subsequently supported by Sveshnikova's experiments [32] with copper-brass and duralumin specimens. If the principle of continuity is to be considered valid, then these results indicate the conicity of the loading point. At the same time they confirm the effect of a sharp break in the deformation path at orthogonal loading.

Feigen [33] describes his experiments, conducted on aluminum alloy specimens with very small torsional moments in relation to large tensile forces. The dependence of the instantaneous shear modulus on the tensile force was observed; however, the deviation from the elastic value was considerably smaller than in the two papers just mentioned. Feigen's experiments are interesting since they confirm directly the occurrence of the breaking effect of the plastic deformation trajectory.

Comparing experimental data with deductions of the flow theory (Laning's law) and of the deformation theory, Feigen concludes that the experimental data for stepwise loading which he performed, lie between the values obtained on the basis of those theories.

Important are two experimental papers by Hu and Marin. In the first [34], the behavior of a loading surface in the plastic deformation process is studied. The authors ascertained that during this process changes of the loading surface cannot be accounted for by mere isotropic expansion or rigid displacement. It is not hard to see that this conclusion rejects the principle of independent action of every one of the numerous loading surfaces (this is particularly clear on the example of plane loading surfaces).

In the second paper by Hu and Marin [35] special attention is given to plastic deformations for a special loading path, which, for purposes of brevity, we will call circular. It is followed in this manner: the material is brought into the plastic range, then (beginning of circular loading) the stress vector describes, on some plane of the vectorial space, an arc of a circle

$$|\mathbf{P}| = \text{const}$$

For any plasticity theory in which the condition $dI_2 > 0$ is the loading criterion, circular loading is neutral, and a change of plastic deformation should not take place along this path. Contrarily to this, experiments have shown that at circular loading a change in the plastic deformation takes place which is so considerable that it cannot be explained by inaccuracy of the experiment or by anisotropy. From the data

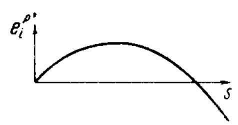

Fig. 6.

cited in the paper, the following qualitative picture of plastic deformation change at circular loading is obtained: if S is a measure of the length of the path traced by the tip of the stress vector, and e_i^{p*} is the increase of the intensity of plastic deformations above the constant intensity which was achieved at the beginning of the circular loading, then the specific form of the relation between e_i^{p*} and S may be represented as follows (see Fig.6). It should be noted that the Hu and Marin experiments encompass all states of biaxial tension, from simple tension in one direction to simple tension in the other direction.

The second paper of Hu and Marin also attempts to determine experimentally a portion of the loading surface (more correctly, curve, since a plane loading path is considered) for the given point of the loading path. It was proved that this part of the loading surface is close to the one which is predicted by the slip theory.

Summarizing the results obtained from experimental research, it should be said that until now only insufficient results that are quantitative and reliable were obtained with respect to change of surface loading, breaking of plastic deformation curve, conicity of the loading point, etc., so that intensive research in those directions is warranted. This is the reason why we were attempting, in this paper, to avoid a discussion of the quantitative aspects, and it might be said that the quantitative factor varied in the experiments of various authors to an extent that seems paradoxical at the modern level of experimental techniques. This is especially applicable to the determination of the instantaneous shear modulus.

Things look much better with respect to the qualitative side of the matter. It seems to us that there is every reason to believe already now that the instantaneous shear modulus differs from the elastic modulus in the plastic range and depends on the magnitude of the stress at which orthogonal loading was effected. Related to it is the existence of the breaking effect of the plastic deformation curve at orthogonal loading. There is no doubt that at circular loading, a change of plastic deformation takes place.

Even if we were to assume that the presently available data are not sufficient to definitely assert the existence of a conical loading point, nonetheless the concept of the conical point is almost the only hypothesis for the explanation of the above-mentioned phenomena.

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